# A NON-TECHNICAL INTRODUCTION TO LINEAR PROGRAMMING AND ITS USE IN RESERVOIR OPERATIONS

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# A Non-technical Introduction to Linear Programming and Its Use in Reservoir Operations

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The object of this paper is to take the mystery out of the term "Linear Programming" (LP), particularly as applied to reservoir operations. It is designed for readers unfamiliar with LP. It should provide the background and understanding required to participate fully in discussions of linear programming techniques during the National Workshop on Reservoir Systems Operations. While it is not a comprehensive review, a selected bibliography is included.

## What is Linear Programming?

Linear Programming is a mathematical technique very similar to the methods used to solve simultaneous equations. (Remember two equations in two unknowns or three equations in three unknowns from algebra?) LP allows its user to "examine all possible solutions" to a problem, and to find the solution which "best" fits a particular objective or a weighted mix of objectives. "Best" is in quotes because the problem being solved is, in most cases, an approximation of the real problem. The resulting solution is not really the "best," therefore, but rather an approximation of the best solution to the real problem.

## How Do You Use Linear Programming?

In order to use LP, the problem to be "solved" must be described in a very specific and comprehensive manner. This is often called the "format" of an LP problem.

The format has two distinct parts, first an "objective function," and second a set of constraints. Both are linear combinations of variables, similar to equations. The objective function is a mathematical description of the good or service which is to be maximized or the cost or impact to be minimized. The constraints are mathematical descriptions of things which must be true if a solution is to be realistic (feasible).

The concepts in the preceeding paragraph should be much clearer given an example. Consider the case of maximizing the release from a reservoir over a historical period 120 months long. If $X_i$ is the release from the reservoir in month i, one objective might be to maximize the sum of the releases in each month. The corresponding objective would be written:

Maximize $X_1 + X_2 + .....+X_{119} + X_{120}$

or

Maximize $\Sigma X_i$

$i=1, 120$

In order for any set of releases to be feasible, they must not exceed the water available to be released in any of the months. This can be written as a set of equality constraints; one for each month i,

$$X_i = S_i - S_{i+1} + R_i - W_i$$

where

$X_i$ = Release in month i (unknown)

$R_i$ = Inflow in month i (known from historical record)

$S_i$ = Storage at the beginning of month i (unknown)

$W_i$ = Spill in month i (unknown)

Each of these constraints simply states that the release in month i must equal the change in storage during the month, plus the inflow during the month, minus the spill during the month. These constraints are called the "continuity equations for the reservoir."

Thus,

$$\text{Max} \quad \sum_{i=1,\ 120} X_i$$

subject to

$$X_i = S_i - S_{i+1} + R_i - W_i \quad \text{for } i=1,\ 120$$

$$X_i,\ S_i,\ W_i \geq 0$$

is a linear programming problem, with an objective function and a set of constraints.

Any linear programming problem can be described in format very similar to the problem above. The general format is:

$$\text{Max} \quad C_1X_1 + C_2X_2 + \ldots\ldots + C_mX_m$$

subject to

$$A_{11}X_1 + \ldots + A_{i1}X_i + \ldots + A_{m1}X_m \leq B_1$$

$$\vdots$$

$$A_{1j}X_1 + \ldots + A_{ij}X_i + \ldots + A_{mj}X_m \leq B_j$$

$$\vdots$$

$$A_{1n}X_1 + \ldots + A_{in}X_i + \ldots + A_{mn}X_m \leq B_n$$

$$X_1 \ldots\ldots X_n \geq 0$$

for a problem with m variables and n constraints. In this format, the $C_i$ and $A_{ij}$ are known coefficients; the $B_j$ are also known, and the $X_i$ are unknowns called "decision variables". The reservoir problem above can be put in this format:

$$\text{Maximize } 1X_1+1X_2+...+1X_{120}$$

$$\vdots$$

$$1X_1+0X_2+0X_3+....-1S_1+1S_2+0S_3+....+1W_1+0W_2+... = R_1$$

$$\vdots$$

$$0X_1+..+1X_i+..+0S_1+..-1S_i+1S_{i+1}+..+0W_1+..+W_i+..= R_i$$

$$\vdots$$

$$.......1X_{120}+......-1S_{120}+1S_{121}+.......+1W_{120} = R_{120}$$

(Note that in the general LP format, there are no powers, exponentials, or products of decision variables. All the terms in the format are either constants or the product of a variable and a constant, hence the name Linear Programming).

<u>How are Linear Programs Solved?</u>

The following is intended to give the reader a feel for what goes on in a computer when an LP is being solved. Once the problem is organized in an LP format, a canned computer program is generally used to obtain solutions. These programs are often easy to use and well documented. In fact, the most difficult part of using LP is rarely using the computer to obtain a solution. The process of understanding a problem and then reducing it to the LP format without oversimplifying it usually involves far more time and effort than setting up the program to solve it. In addition, data gathering is often more difficult than formulation and solution put together.

Nonetheless, it is helpful to have a basic understanding of how the canned programs work.

For simplicity, assume the problem above only considered two time periods. The constraints for the two period problem are:

$$X_1 - S_1 + S_2 + W_1 = R_1$$
$$X_2 - S_2 + S_3 - W_2 = R_2$$

Assuming that $S_1$ and $S_3$ (the initial storage and ending storage) are both fixed, this is a set of two simultaneous equations in five unknowns, $X_1$, $X_2$, $S_2$, $W_1$, $W_2$. The reader may remember from algebra that there will be an infinite number of solutions to such a set of equations. But, if any three of the variables are set to zero, then there remains a set of two equations in two unknowns, and these can easily be solved. Further, the results of that solution can be plugged into the objective function and an objective value corresponding to that solution can be obtained. In this five variable, two equation problem there are ten distinct pairs of variables for which the equations can be solved. One of the major theorems of LP states that one of these solutions is optimal, that is, gives a value for the objective function as high as or higher than any other of the infinite number of solutions to the equations.

Again, the present problem could be solved for each of the ten pairs of variables, the objective function values compared, and the optimum pair selected. This is called enumeration. For larger problems, this is not so easy. For example, a problem with 20 unknowns and ten constraints has over 46,000 unique sets of ten variables. Enumeration, even for a relatively small problem, can be quite tedious.

LP solution techniques find the optimal combination of variables without enumerating. Instead, the techniques start with one combination and then examine all combinations which differ from that combination by only one variable. If one of the new combinations provides a better value for the objective function, then it is substituted for the original. The process is repeated until a combination is found that has a better value of the objective function than any of the combinations which differ from it by only one variable. A second major theorem of LP states that such a combination is indeed optimal.

That, in a nutshell, is how a computer solves an LP. The actual programs make use of quite sophisticted matrix solution techniques, but the end result is the same.

## What Makes LP Formulations Realistic?

There is, unfortunately, no easy answer to this question. About the only way to shed light on this topic is by illustration. Starting with the two period example discussed earlier, one possible optimal solution to the problem as stated has variables $S_2$ and $X_2$ positive. For this solution:

$$S_2 = S_1 + R_1 \quad \text{and}$$
$$X_2 = S_1 - R_1 - S_3$$

(Remember, $S_1$, $S_3$, $R_1$, and $R_2$, the Initial Storage, Ending Storage, and Inflows during months 1 and 2 are all known in advance). Suppose that the reservoir was full at the beginning of the first month, that is $S_1$ equalled the capacity of the reservoir. If that is so, than this solution is unrealistic, for the reservoir can't hold as much water as would be required at the beginning of month 2, that is $S_1$ (full) plus $R_1$ (the inflow during the month 1).

To eliminate this problem, a constraint on $S_2$ can be added:

$$S_2 \leq S^*$$

where $S^*$ = the capacity of the reservoir. (In the 120 month problem, 119 constraints would be added, $S_i \leq S^*$ for i from 2 to 120. $S_1$ is previously known).

Now consider another potential optimal solution to the two period problem, one with $X_1$ and $X_2$ positive. In this solution:

$X_1 = R_1 + S_1$ and

$X_2 = R_2 - S_3$

This solution says to empty the reservoir during period 1, and then release only what is not needed to refill the reservoir during the second period. (Note the if $R_2$ is less than $S_3$, this solution is not feasible since $X_2$ would be negative). Such a pattern of release is generally not desirable, even though it may maximize the total amount of water released. To correct this, the objective function can be redefined. Generally, reservoirs are operated to even out the flow of a stream, and the term "yield" is often applied to the maximum even release that a reservoir could provide over a historical record. Define a variable $X^*$ (Note: $X^* = X_i$ for all i, since releases are even) to be the even monthly release from the reservoir, and make the new objective Maximize $X^*$.

Using the new objective (and substituting $X^*$ for the $X_i$ in the constraints), and adding the constraints on storage capacity, gives the following LP for the 120 period problem:

Maximize $X^*$

subject to

$$X^* - S_i + S_{i+1} + W_i = R_i \quad \text{(for i=1, 120)}$$

$$S_i \leq S^* \quad \text{(for i = 2, 120)}$$

$X^*$, $S_i$, $W_i$ all positive.

This formulation is substantially more realistic than the original, and when it is solved, $X^*$ is the oft referred to theoretical "safe yield" of the reservoir.

In sum, creating realistic LP formulations of problems is as much an art as it is a science. It requires a thorough understanding of the problems at hand, and the data available. A willingness to reexamine the formulation in light of the kinds of problems which might be encountered during implementation is also essential.

Other Examples of LP Objectives and Constraints

It is quite easy to incorporate seasonal or monthly releases in an LP formulation. An appropriate coefficient is simply added to the $X^*$ in the constraints:

$$A_iX^* - S_i + S_{i+1} + 1W_i = R_i$$

where $A_i$ is the percentage of the average monthly release ($X^*$) desired in month i. The $A_i$ can be repeated every 12 months, so that $A_1=A_{13}=A_{25}$, and $A_2=A_{14}=A_{26}$ etc.

Similarly, provision for flood control can be made by modifying the constraints on maximum storage:

$$S_i \leq S^* - F_i$$

where $F_i$ is the free storage required for flood control in month i. $F_i$ can be set such that the expected recurrence of floods larger than a set size is greater than a set number of years. This determination can be made using standard hydrologic information before the LP is run. If $F_i$ is set to maintain such a limited probability of flooding, than this constraint may be referred to as an "implicit stochastic" constraint.

Minimum storage for recreation can be required:

$$S_i \geq M_i$$

where $M_i$ is the minimum storage required in month i.

Evaporation can be added to the continuity equations in the LP using a technique called piecewise linearization. Figure 1 is typical graph of evaporation as a function of storage in a reservoir.

The curve can be broken up into sections, 1 - 3, represented by the lines in Figure 1. $g_1$ and $g_2$ are the values of storage at the points where the segments meet. $G_j$ is the slope of segment j, and $s_{ij}$ is its length along the storage axis. Now $E_i$, the evaporation in month i can be defined:

$$E_i = G_1 s_{i1} + G_2 s_{i2} + G_3 s_{i3}$$

with $s_{i1} \leq g_i$

$s_{i2} \leq g_2 - g_1$

$s_{i3} \leq S^* - g_2$

and $S_i = s_{i1} + s_{i2} + s_{i3}$

When these constraints are included in the LP information, the continuity equations can be rewritten to account for evaporation:

$$X^* - S_i + S_{i+1} + W_i + E_i = R_i$$

Piecewise linearization is a particularly powerful tool, since it allows LP's to consider some non-linear relationships. It can be used with functional transformations. Hydropower production from a single

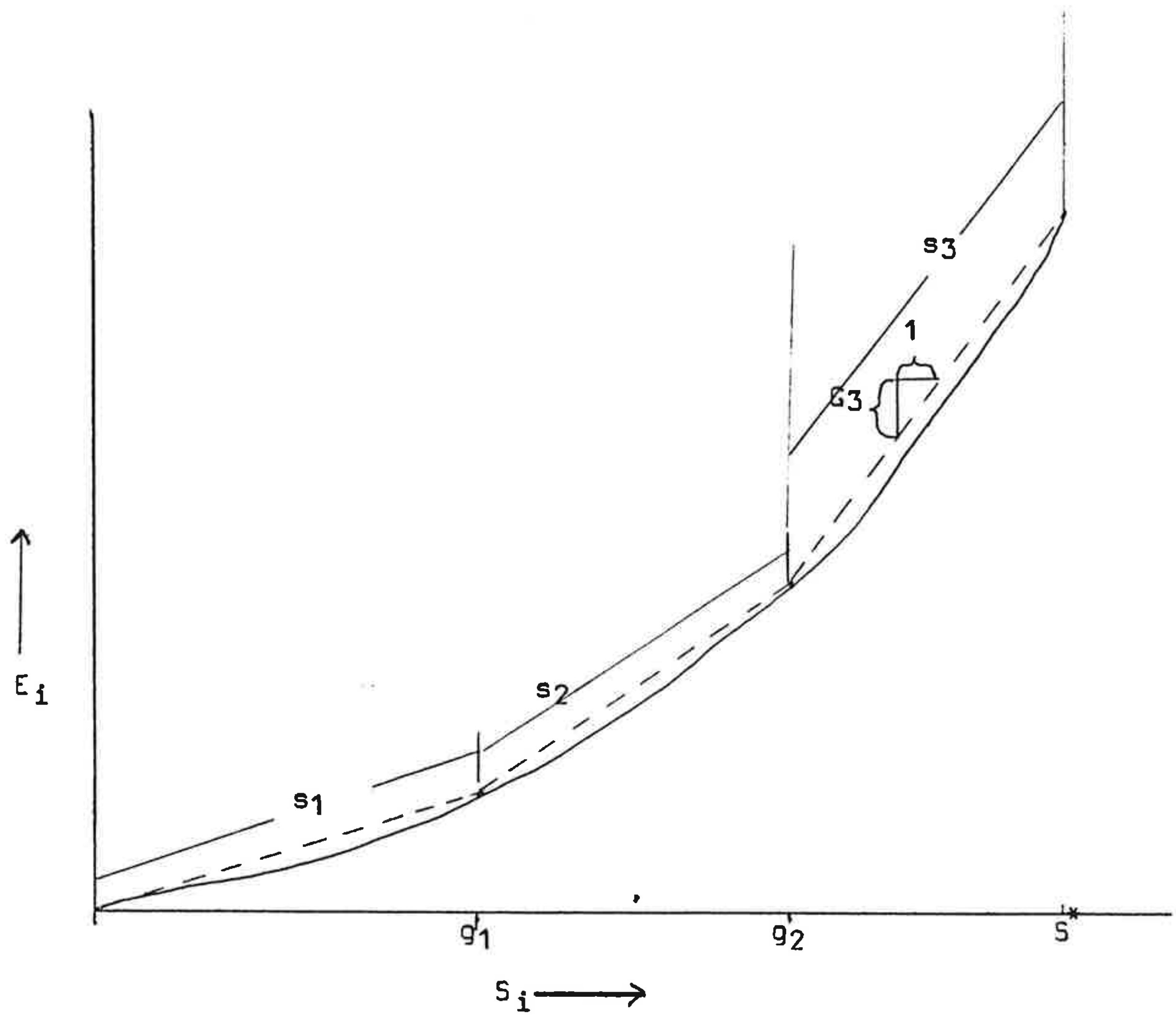


Figure 1  Piecewise Linearization of Evaporation Function

reservoir can be required to be above a certain level using this combination of techniques. Power produced in month i is proportional to the product of head, $H_i$ and release $X_i$. Using a proportionality constant p and defining $P_i$ as the minimum power required in month i, the power constraints can be written:

$$pH_iX_i \geqq P_i$$

Since this contains the product of two decision variables, head (a function of storage), and release, it cannot be used directly in an LP. However, logarithms can be taken (a functional transform) to obtain an equivalent condition:

$$\ln(p)+\ln(H_i)+\ln(X_i) \geqq \ln(P_i)$$

$\ln(H_i)$ can be piecewise linearized as a function of storage $S_i$ (just as with evaporation $E_i$ above), $\ln(X_i)$ can be piecewise linearized as a function of release $X_i$, and the condition can then be included in an LP. (p and $P_i$ are constants known in advance, and so their logs are also constants.)

One common "flaw" in all the formulations so far discussed is that they assume perfect foreknowledge of inflows to the reservoir, and then determine the optimal operating scheme. To avoid this, releases may specified to be according to a rule based on reservoir storage and previous inflows. The LP may then be used to determine the optimal parameters for these rules. (The formulations below are based on work by ReVelle and coworkers. See references.) Defining $Z_i$ as a factor applied to the previous months inflow to determine releases, and $B_i$ as a variable to be set in the LP, a linear decision rule for releases can be written:

$$X_i = S_i - B_i + Z_i R_{i-1}$$

When this constraint is included in a formulation, the LP will solve for the $Z_i$ and $B_i$ which, when used in the above release rule, would have maximized the objective over the historical record. The rule could then be followed as a guide in the future.

The constraints developed above can be used in conjunction with many objectives. $X^*$ could be taken as known, $S^*$ left as a variable and the objective set as Minimize $S^*$. If both $X^*$ and $S^*$ are set, power production could be maximized from a single reservoir. Weighted sums of the objectives can be maximized to determine the tradeoffs between objectives. (This is called linear multi-objective programming). Other possible objectives include minimizing deviation from monthly target releases and maximizing economic benefits as a function of monthly releases.

## What are the major limitations on the use of LP?

The two major limitations on the use of LP are 1) the fact that not all problems can be fitted to an LP format, and 2) a limit on the size of the problem which can be solved in a modern computer. The difficulty in solving problems on computers is most directly related to the number of constraints in the problem. Problems as large as several thousand constraints can currently be solved, but solutions may be expensive. This restricts the use of LP generally to investigations of monthly operations over larger periods, or to weekly, daily or hourly operations over much shorter periods.

Fitting problems to an LP format, however, is an art as much as it is a science. Some things, like a hydropower constraint for multiple reservoirs (analogous to the previous hydropower constraint):

$$p_1 H_{i1} X_{i1} + p_2 H_{i2} X_{i2} + \ldots\ldots \geq P_i$$

where the numerical subscript refers to the different reservoirs, simply cannot as yet be formulated in an LP. Other major problems occur when variables must be discrete, i.e., a release has no value unless it fully satisfies a particular demand, or a decision must be made to either not build a facility or to build it at a minimum size. These problems can be overcome by using a technique called mixed integer programming which requires a format essentially the same as that for LP with the exception that discrete variables are allowed. The problem then is the large reduction in the size of the problems which can be solved.

## Summary and Conclusions

So far, LP format, and the basic method of solution of LP problems have been described, and various simple applications of LP to reservoir operation problems have been developed. Some of these used tricks of the LP trade, piecewise linearization and functional transforms. There has been a brief discussion of linear operating rules and also of some of the limitations of LP. In concluding, some further discussion of the basic nature of linear optimization is in order.

Natural processes are rarely truly linear. Therefore, any attempt to describe them in the linear fashion required by LP forces an approximation. The optimal solutions to LP problems are, therefore, only approximations of optimal solutions to real problems. Further, in developing LP formulations, important constraints can be overlooked or very poorly approximated. This can lead to nonsense solutions. The best measure of the appropriateness of an LP formulation is the utility of the solutions it provides in guiding real operations. Every LP solution must be evaluated in this light.

On the other hand, LP can be and has been a valuable tool in improving reservoir operations, as case studies at the National Workshop on Reservoir Systems Operations will show. The very act of formulating an LP problem forces an assessment of objectives, of the relationship between operations and these objectives, and of the data needed to address the problem. Solutions to LP problems can provide added insight and new and better operating policies.

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